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On the Derivation of Thermohydrodynamic Equations from the Boltzmann Equations

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The Boltzmann equation deals with a distribution $f(x, \xi)$, where x denotes the space variable and ξ is the momentum. The hydrodynamic equations deal with ξ-moments of the distribution. The paper deals with the derivation of the hydrodynamic equations in the case that the collision kernel is Maxwellian, i.e., independent of the velocity. For such a kernel, a computational tool, based on the theory of representations of the orthogonal group, is developed. With this tool it is possible to derive systems of equations for any number of moments. The construction of closed systems is based on asymptotic estimates for solutions of Boltzmann equations. These show that, in some definite sense, an approximating system involving moments of high order is more accurate than a system of lower order.

KEY WORDS: Boltzmann equation; hydrodynamic equations; collision kernel; Maxwellian molecules; representation of the orthogonal group; moments; closure problem; asymptotic estimates.

1. INTRODUCTION

In 1865 James Clerk Maxwell published the fundamental paper, "On the Dynamical Theory of Gases."[12] In this paper he was interested in the computation of diffusion and viscosity coefficients and, mainly, in the heat conductivity of gases. The existence of diffusion, viscosity, and heat conductivity previously had been postulated in hydrodynamics, but Maxwell devised a grand plan to derive the equations of hydrodynamics from basic principles. He thus created the field of statistical mechanics or, more precisely, of what is now called kinetic theory. The paper deals with the connection between kinetic theory and hydrodynamics.

Maxwell considered an ensemble of molecules that are marked by space (x) and momentum (ξ) coordinates [six-dimensional space (x, ξ)]. This

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ensemble is subject to change in time due to mutual interaction between the molecules. If the interaction is known, one can, at least in principle, integrate the equations in 6-space and time. To solve the problem for a general law of interactions was (and still is) beyond the state of the art in mathematics, so Maxwell was led to postulate some "simplifying assumptions."

He assumed a short-range binary potential acting between pairs of molecules. By letting the range of this potential tend to zero and by assuming density $f(x, \xi)$ instead of discrete distribution, the force action due to the short-range binary potential was converted to time change of $f(x, \xi)$ only in ξ space. (The famous "molecular chaos" postulate is part of the above assumption.)

Maxwell was interested mainly in computing the time change of integrals of $f(x, \xi)$ in ξ space (such as average momentum, energy, and some moments of higher order). In order to carry out an explicit calculation, he assumed a very particular potential, one that makes the resulting force proportional to the inverse fifth power of the distance. With this force he was able to calculate explicitly the time change of some of the moments (in ξ space) of $f(x, \xi)$, x being kept constant. Moreover, he could also show that these moments have limit values (as t tends to infinity).

Maxwell considered the full motion in (x, ξ) space and decomposed it into streaming $[f(x, \xi, t) = f(x - \xi t, \xi, 0)]$ and "collision" (i.e., the action of the short-range binary potential). Then he wrote down equations of transfer for the moments. He assumed that the "collision" process is much faster than streaming. Consequently, he discarded certain terms, inserted limits to others, and arrived at expressions for the diffusion and viscosity coefficients as well as Euler and Navier–Stokes equations.

The molecular-chaos, binary-collision model is widely accepted for dilute (nonpolar) gases as a "realistic" one. Boltzmann and later Grad[6,17] showed, by formal arguments, that the celebrated Boltzmann equation is obtained from the binary collision model by a limit process by which the size of the molecules tends to zero and the number tends to infinity such that the total occupied volume is held fixed.

Lanford,[13] in a recent publication, sketched a procedure by which the Boltzmann equation is obtained from the BBGKY hierarchy. The key assertion is that the validity of the molecular chaos assumption for $t = 0$ implies, under some restrictions, its validity for a small, positive time.

Maxwell was aware of the Boltzmann equation. He thought, though, that the only equations having physical significance are the equations derived for the moments (his "equations of transfer"). (In a later paper, in 1879, "On Stresses in Rarified Gases Arising From Inequalities of Temperature," he himself referred to the Boltzmann equation.)

Boltzmann's model even now is far from being justified mathematically and it is not clear what the exact passage is from finite, discrete distributions to the continuous limit.

This paper goes in another direction. Maxwell was not precise in his sequence of approximations and truncations. In his reasoning he combined formal expansions and physical intuition. Improvement of his procedure has to start with an exact definition of the formal expansion. Hilbert suggested a formal power series expansion (cf. Ref. 17). Chapman and Erskog (working separately) obtained, essentially by Hilbert's expansion, the coefficients of viscosity and conductivity for fairly general molecules (cf. Ref. 3). Grad[6] and Truesdell[11] suggested thermohydrodynamic equations based on formal expansions. Grad used expansion in Hermite polynomials; the Truesdell and Ikenberry[10] expansion has a more complicated description. The equations suggested by the various writers do not coincide. Since the formal expansions were hardly supported by asymptotic estimates, it was not possible to decide what the "true" equations are. (For a detailed criticism the reader is referred to Refs. 10 and 11.)

The main objection to Maxwell's procedure is to the use of the mysterious molecules bearing his name. An improvement was required in the direction of the treatment of a general, or at least a "realistic," law of force. Hardly any advance has taken place in that direction.

This paper stays close to Maxwell. A "collision kernel" is substituted for the force law. The basic assumption is that the "collision" (i.e., the action of the short-range potential) is independent of the relative (scalar) velocity of the two particles. (Maxwell's r^{-5} law is a particular case.) Accordingly, these molecules will be called "generalized Maxwellian molecules." Then the structure of the collision operator is studied. The theorems established are extension of theorems due to Ikenberry.[10] The method developed in this paper is different. It is based on the powerful tools of representation theory of the orthogonal group. The proofs are, even in the Maxwellian case, much simpler than those of Ikenberry.

The Maxwell–Boltzmann equation for the spatially homogeneous case (generalized Maxwellian molecules) is studied first. A theorem of convergence for the ξ-moments of the distribution $f(\xi, t)$ is proved.

The theorem states that as t tends to infinity, all the moments of the distribution $f(\xi, t)$ tend to polynomials in the basic five moments: density, momentum, and energy. This theorem, in a weaker form, is known for a more general case. In the case of Maxwellian molecules, it coincides with the corresponding theorem in Ref. 10.

Anticipating the problem of the construction of hydrodynamic equations in the spatially inhomogeneous case, we turn to the estimate of the moments of a certain order by moments of a lower order (not necessarily the basic five invariants). It is shown that every moment approaches a polynomial in lower order moments. Moreover, it is shown that the estimates get better as the order of the moments gets higher.

The inhomogeneous Maxwell–Boltzmann equation is dealt with in the final sections. Equations for ξ-moments of the distribution $f(x, \xi)$ are derived. These equations involve the time derivative of a certain moment and space (x) derivatives of moments having higher order. At this point we suggest how to "close" the equation and get thermohydrodynamic equations. The recipe suggested is simple (at least in principle) and straightforward. The reasoning is based on basic well-known arguments which are, unfortunately, heuristic. In the Boltzmann equation (1), the term $\sum_{i=1}^{3} \xi_i \, \partial f/\partial x_i$ represents streaming, i.e., change of position, with the momentum kept fixed. The term $J(f, f)$ represents collisions, or the action of a short-range binary potential. Thermohydrodynamic equations are to be obtained by a limiting process where the collision term becomes larger and larger with respect to the streaming term. Suppose, consequently, that a parameter $1/\epsilon$ is inserted in front of the collision term $J(f, f)$. Choose some point x_0 and compare the spatially inhomogeneous equation around x_0 to the spatially homogeneous equation where the distribution $f(x, \xi)$ takes the value $f(x_0, \xi)$. For the spatially homogeneous case, if ϵ is small, after a short interval of time the ξ-moments of the distribution will be close to their limiting expressions (a proof is furnished). For the spatially inhomogeneous equation, if ϵ is small indeed, the streaming term will not change very much the relations between the ξ-moments of the distribution. We suggest, therefore, to substitute high-order moments by limiting expressions consisting of polynomials of lower order moments. This will "close" the equations. The estimates obtained in Section 5 show, in some sense, that closing the equations at a higher level is better.

Some of the collision kernels do not have a finite cross section (Maxwell's law of force leads to such a kernel). In these cases the kernel is approximated by cutoff kernels having a finite cross section, and then a limit is taken. In this way kernels that grow like $\theta^{-3+\epsilon}$ (as $\theta \to 0$) can be treated. It will be seen in an appendix that Maxwell's kernel has a singularity of order at most $\frac{3}{2}$.

A remark about the generality of the results. As was stated before, the results established hold for "generalized Maxwellian molecules." In this case the spatially homogeneous Maxwell–Boltzmann equation is directly reduced to equations in the moments. These equations are invariant under rigid transformations. Furthermore, they preserve positivity of density, energy, etc. If one starts with a so-called "general law of force," then, even in the spatially homogeneous case, an attempt to set up equations for the moments has to involve truncations, linearizations, etc. These truncations may not keep the necessary physical properties of the various moments. Our view is that one should prefer a particular law of force, provided that physically meaningful equations can be obtained. (A fine discussion of "general" versus particular laws of force is contained in Ref. 1.)

2. THE MAXWELL–BOLTZMANN EQUATION

The material in this section is introduced for two reasons: as a short reminder of the classic derivation and also for the purpose of familiarizing the reader with the notation.

The celebrated Boltzmann equation governs a distribution $f(x, \xi, t)$, where $x = (x_1, x_2, x_3)$ denotes the space variable and $\xi = (\xi_1, \xi_2, \xi_3)$ denotes the momentum variable. $f(x, \xi, t)$ is considered to be a limit of a finite ensemble of molecules in 6-space. The equation is

$$\frac{\partial f}{\partial t}(x, \xi, t) + \sum_{i=1}^{3} \xi_i \frac{\partial f}{\partial x_i}(x, \xi, t) = J(f,f) \tag{1}$$

The form $J(f,f)$ already assumes molecular chaos.

The term $\sum_{i=1}^{3} \xi_i\, \partial f/\partial x_i$ represents the time change due to streaming. $J(f,f)$ represents the change due to collisions. It is a quadratic operator and is invariant in any subspace $x = x_0$ [i.e., $J(f,f)(x_0, \xi, t)$ depends only on $f(x_0, \eta, t)$, where η varies in a three-dimensional momentum subspace]. Let us exhibit its structure. For that purpose one may omit the x, t variables. In the case of a finite, discrete distribution of molecules the existence of a short-range binary potential between pairs of molecules is assumed. It is convenient to use the term "collision" in order to describe the action of such a potential. The details of the collision process are of no interest, only the outcome (a situation similar to the theory of scattering). It is assumed that collisions are elastic. Let ξ and η be incoming velocities and $\bar{\xi}$ and $\bar{\eta}$, respectively, the outgoing velocities. Then

$$\begin{aligned} \bar{\xi} &= (1/2)(\xi + \eta) + (1/2)\zeta|\xi - \eta| \\ \bar{\eta} &= (1/2)(\xi + \eta) - (1/2)\zeta|\xi - \eta| \end{aligned} \tag{2}$$

where ζ is some random vector, $|\zeta| = 1$, distributed on the unit sphere. Its distribution is dependent on the mechanism of collision. It can be assumed that it is given explicitly. (This assumption "covers" the "molecular chaos" assumption or the postulate of probabilistic laws of nature, etc.) The distribution function of ζ may depend on $|\xi - \eta|$. The invariance with respect to rigid motions prevents a different type of dependence on ξ and η. Taking into account the indistinguishability of the two molecules, one is led to:

Hypothesis 2.1. There exists a kernel $K_0(\theta, |\xi - \eta|)$, where θ is the angle between ζ and $\xi - \eta$, so that the probability that ζ forms an angle θ

($\theta_1 < \theta \leqslant \theta_2$) with $\xi - \eta$ is $K_0(\theta_2, |\xi - \eta|) - K_0(\theta_1, |\xi - \eta|)$. The measure dK_0 is positive and is a function of $\sin\theta$ [i.e., $dK_0(\theta, |\xi - \eta|) = dK_0(\pi - \theta, |\xi - \eta|)$].

Let us turn back to (1). The vector $\zeta \cdot |\xi - \eta|$ can be expressed as

$$\zeta \cdot |\xi - \eta| = (\cos\theta)(\xi - \eta) + (\sin\theta)\zeta^* |\xi - \eta| \tag{3}$$

where ζ^* is a unit vector perpendicular to $\xi - \eta$. Since K_0 is dependent only on θ and $|\xi - \eta|$, it follows that the collision law also can be expressed by

$$\begin{aligned} \bar{\xi} &= (1/2)(\xi + \eta) + (\cos\theta)(1/2)(\xi - \eta) + (\sin\theta)(1/2)\zeta|\xi - \eta| \\ \bar{\eta} &= (1/2)(\xi + \eta) - (\cos\theta)(1/2)(\xi - \eta) - (\sin\theta)(1/2)\zeta|\xi - \eta| \end{aligned} \tag{4}$$

where ζ is a unit vector evenly distributed on a plane perpendicular to $\xi - \eta$.

The collision law of two molecules leads to a law for the rate of change of a positive measure (mass) $f(\zeta)$, where $f(\zeta)$ denotes the limit of a distribution of a finite number of molecules. The collision law (4) and the molecular chaos assumption lead to the well-known form

$$J(f,f)(\xi) = \frac{1}{2\pi} \iiint [f(\xi')f(\eta') - f(\xi)f(\eta)]\, dK(\theta, |\xi - \eta|)\, d\varphi\, d\eta \tag{5}$$

where

$$\begin{aligned} \xi &= (1/2)(\xi' + \eta') + (1/2)\zeta'|\xi' - \eta'| \\ \eta &= (1/2)(\xi' + \eta') - (1/2)\zeta'|\xi' - \eta'| \end{aligned} \tag{6}$$

θ is again the angle between ζ' and $\xi' - \eta'$. Here φ is the polar angle in a plane perpendicular to $\xi - \eta$. The measure dK incorporates the effect of the rate of collisions (as well as the outcome of each collision), which is dependent, generally, on $|\xi - \eta|$. Therefore $dK \neq dK_0$ in the general case.

Elastic collisions of hard balls lead to $dK(\theta, |\xi - \eta|) = c|\xi - \eta| \sin\theta\, d\theta$; Maxwell's law of force leads to $dK(\theta) \leqslant (K_1 + K_2\theta^{-3/2})\, d\theta$ (cf. the appendix). The kernel has infinite cross section.

Let $f(\xi)$ denote mass distribution (positive measure), for which all the moments exist. Denote moments by ψ_β, where β is a multiindex: $\beta = (\beta_1, \beta_2, \beta_3)$. The first moment in the ξ_1 direction will be denoted by $\psi_{(1,0,0)}$:

$$\psi_{(1,0,0)} = \int \xi_1\, df(\xi) \tag{7}$$

Denote by ξ^β the product $\xi_1^{\beta_1}\xi_2^{\beta_2}\xi_3^{\beta_3}$. Thus

$$\psi_\beta = \psi_{(\beta_1,\beta_2,\beta_3)} = \int \xi_1^{\beta_1}\xi_2^{\beta_2}\xi_3^{\beta_3}\, df(\xi) = \int \xi^\beta\, df(\xi) \tag{8}$$

ψ_β will have the order $|\beta| = \beta_1 + \beta_2 + \beta_3$. The ψ_β are dependent on $f(\xi)$. This dependence will be omitted.

In order to be closer to physicists' notation, $df(\xi)$ will be replaced by $f(\xi)\, d\xi$, i.e., arguing as if a density function does exist. The density assumption will never be used and the interested mathematician can routinely restore $df(\xi)$, measures, Stieltjes integration, etc., to the theorems and proofs. By the same token, $dK(\theta, |\xi - \eta|)$ is replaced by $K(\theta, |\xi - \eta|)\, d\theta$.

Let us write

$$\omega_\beta = \int \xi^\beta J(f, f)\, d\xi \tag{9}$$

Lemma 2.1. Let $K(\theta, |\xi - \eta|)$ have a finite cross section [i.e., $\int K(\theta, |\xi - \eta|)\, d\theta < \infty$]. Then

$$\begin{aligned}\omega_\beta = (1/2\pi)(1/2)^{|\beta|} \iiiint & [\xi + \eta + (\cos\theta)(\xi - \eta) + (\sin\theta)|\xi - \eta|\zeta]^\beta \\ & \times K(\theta, |\xi - \eta|) f(\xi) f(\eta)\, d\varphi\, d\theta\, d\xi\, d\eta \\ & - \int \left[\iint K(\theta, |\xi - \eta|) f(\eta)\, d\theta\, d\eta\right] \xi^\beta f(\xi)\, d\xi\end{aligned} \tag{10}$$

where $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$. ζ is a unit vector in a plane perpendicular to $\xi - \eta$ and is evenly distributed in φ.

Proof. By (5) and (6),

$$\begin{aligned}\omega_\beta &= (1/2\pi) \iiiint \xi^\beta [f(\xi') f(\eta') - f(\xi) f(\eta)] K(\theta, |\xi - \eta|)\, d\theta\, d\varphi\, d\eta\, d\xi \\ &= \iiiint [\tfrac{1}{2}(\xi' + \eta') + \tfrac{1}{2}\zeta' |\xi' - \eta'|]^\beta \\ &\qquad \times f(\xi') f(\eta') K(\theta, |\xi - \eta|)\, d\theta\, d\varphi\, d\eta\, d\xi \\ &\qquad - \int f(\xi) \left[\iint K(\theta, |\xi - \eta|) f(\eta)\, d\theta\, d\eta\right] \xi^\beta\, d\xi\end{aligned} \tag{11}$$

The second term is already in the required form. As for the first term, consider for fixed ζ' the transformation $(\xi', \eta') \to (\xi, \eta)$. The Jacobian of this transformation is equal to one. Hence $d\xi\, d\eta = d\xi'\, d\eta'$. Moreover, for elastic collisions $|\xi - \eta| = |\xi' - \eta'|$. Denote, in a way analogous to (3),

$$\zeta' |\xi' - \eta'| = (\cos\theta)(\xi' - \eta') + (\sin\theta)\zeta |\xi' - \eta'| \tag{12}$$

Substitute, now, in the first term of the right-hand side of (11), $|\xi - \eta|$ by $|\xi' - \eta'|$ and $d\xi\, d\eta$ by $d\xi'\, d\eta'$; use also (12) to get the first term on the right-hand side of (10). (The variables ξ', η' are under the integral sign, so they can be converted back to ξ, η.)

Remark. In order that the integral in (11) will not diverge, it is sufficient to assume that $\int K(\theta, |\xi - \eta|)\, d\theta$ is tempered in $|\xi - \eta|$, i.e., its rate of growth at infinity is slower than some power of $|\xi - \eta|$.

The assumption of finite cross section will be retained throughout the paper, up to the end of Section 7. Then it will be seen that the results established hold, by a well-defined limiting process, for kernels having infinite cross sections (those that have a singularity at $\theta = 0$ of order $3 - \epsilon$).

Let us verify, as an exercise in manipulating (11), the invariance of the moments of order one (the momentum) as well as the energy E,

$$E = \tfrac{1}{2}[\psi_{(2,0,0)} + \psi_{(0,2,0)} + \psi_{(0,0,2)}] \tag{13}$$

We have to show that

$$\omega_{(1,0,0)} = \omega_{(0,1,0)} = \omega_{(0,0,1)} = \omega_{(2,0,0)} + \omega_{(0,2,0)} + \omega_{(0,0,2)} = 0$$

Indeed

$$\begin{aligned}\omega_{(1,0,0)} = {}& (1/2\pi)\tfrac{1}{2}\iiiint [\xi_1 + \eta_1 + (\cos\theta)(\xi_1 - \eta_1) + (\sin\theta)|\xi - \eta|\zeta_1] \\ & \times K(\theta, |\xi - \eta|)f(\xi)f(\eta)\, d\theta\, d\varphi\, d\xi\, d\eta \\ & - \iiint \xi_1 K(\theta, |\xi - \eta|)f(\xi)f(\eta)\, d\theta\, d\xi\, d\eta \end{aligned} \tag{14}$$

Perform first the integration $d\varphi$. Since, by symmetry, $\int \zeta_1\, d\varphi = 0$, it follows that in (14) the term containing $(\sin\theta)|\xi - \eta|\zeta$ drops out. All other terms are independent of φ. Integrate now $d\theta$. Since $K(\theta, |\xi - \eta|) = K(\sin\theta, |\xi - \eta|)$, it follows that $\int (\cos\theta)K(\theta, |\xi - \eta|)\, d\theta = 0$.

Therefore the first term on the right-hand side of (14) is reduced to

$$\tfrac{1}{2}\iiint (\xi_1 + \eta_1)K(\theta, |\xi - \eta|)f(\xi)f(\eta)\, d\theta\, d\xi\, d\eta$$

which cancels the second term on the right. Similarly,

$$\begin{aligned}\omega_{(2,0,0)} = \iiint & \{(1/8\pi)\int [\xi_1 + \eta_1 + (\cos\theta)(\xi_1 - \eta_1) \\ & + (\sin\theta)|\xi - \eta|\zeta_1]^2\, d\varphi - \xi_1^2\}K(\theta, |\xi - \eta|)f(\xi)f(\eta)\, d\theta\, d\xi\, d\eta\end{aligned} \tag{15}$$

Upon squaring the inner integral in (15), one gets a term free of ζ. It is constant with respect to φ. The second term will be linear in ζ_1. Since $\int \zeta_1\, d\varphi = 0$, it will drop out. The third term contains ζ_1^2. Let us establish

$$|\xi - \eta|^2(1/2\pi)\int \zeta_1^2\, d\varphi = (1/2)[(\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2] \tag{16}$$

For that purpose denote unit vectors along the coordinate axes by u_i, $i = 1, 2, 3$, and choose a new coordinate system where the unit vectors along the new coordinates will be e_i, $i = 1, 2, 3$. Choose it so that e_3 is parallel to $\xi - \eta$. Then

$$(1/2\pi)\int \zeta_1^2\,d\varphi = (1/2\pi)\int [(\cos\varphi)(e_1, u_1) + (\sin\varphi)(e_2, u_1)]^2\,d\varphi$$

$$= (1/2)[(e_1, u_1)^2 + (e_2, u_1)^2] = (1/2)[1 - (e_3, u_1)^2]$$

Multiply by $|\xi - \eta|^2$ to get

$$|\xi - \eta|^2(1/2\pi)\int \zeta_1^2\,d\varphi = (1/2)[|\xi - \eta|^2 - (\xi_1 - \eta_1)^2]$$

which is equivalent to (16).

It follows by the considerations above that the inner integral in (16) is equal to

$$(1/4)[(\xi_2 + \eta_1) + (\cos\theta)(\xi_1 - \eta_1)]^2 + (1/8)(\sin^2\theta)[(\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2] \tag{17}$$

On adding the corresponding inner integrals from similar expressions for $\omega_{(0,2,0)}$ and $\omega_{(0,0,2)}$, one gets

$$(1/4)\sum_{i=1}^{3} [(\xi_i + \eta_i)^2 + (\xi_i - \eta_i)^2] + (1/2)(\cos\theta)\sum_{i=1}^{3} (\xi_i + \eta_i)(\xi_i - \eta_i)$$

The integration $d\theta$ in (15) is performed next, summed with the corresponding formulas for $\omega_{(0,2,0)}$ and $\omega_{(0,0,2)}$. Since $\int (\cos\theta)K(\theta, |\xi - \eta|)\,d\theta = 0$, it follows that

$$\omega_{(2,0,0)} + \omega_{(0,2,0)} + \omega_{(0,0,2)} = \iiint \left[\tfrac{1}{2}\sum_{i=1}^{3} (\xi_i^2 + \eta_i^2) - \sum_{i=1}^{3} \xi_i^2\right] \times K(\theta, |\xi - \eta|) f(\xi) f(\eta)\,d\xi\,d\eta\,d\theta = 0 \tag{18}$$

3. STRUCTURE OF THE COLLISION KERNEL

The calculation in the preceding section cannot be applied in a direct way to the manipulation of (10). Instead, the powerful tool of the representation theory of the orthogonal group in three-dimensional space will be used (cf. Ref. 5). Reasoning will be based on the following facts:

The orthogonal group admits irreducible representation spaces that are uniquely determined (up to an isomorphism) by a weight l, which is an integer or half an integer. The dimension of an irreducible representation of weight

l is $2l+1$. Thus two irreducible representation spaces having the same dimension are necessarily isomorphic. The unique determination of an irreducible representation space by its dimension holds, for the orthogonal group, only in a three-dimensional space.

The space of spherical functions (i.e., functions defined on the unit sphere) that are square integrable is obviously a representation space. It is infinite dimensional. This space can be expressed as a direct sum of irreducible representation spaces having integral weights. These spaces are composed of spherical functions, the well-known spherical harmonics $Y_l^m(x)$, where l is the weight and $m = -l, \ldots, 0, \ldots, l$. Let ζ be a point on the unit sphere. Denote its spherical coordinates by (θ, φ). Then

$$Y_l^m(\theta, \varphi) = ce^{im\varphi} P_l^m(\cos\theta) \tag{19}$$

where the $P_l^m(x)$ are known as the associated Legendre functions:

$$P_l^m(x) = c(1 - x^2)^{-m/2}\, d^{l-m}(1 - x^2)^l / dx^{l-m} \tag{20}$$

In particular, for $m = 0$,

$$P_l^0(x) = P_l(x) = \frac{1}{2^l \cdot l!} \frac{d^l}{dx^l} (1 - x^2)^l \tag{21}$$

where in (21) the Legendre polynomials $P_l(x)$ are normalized so that $P_l(1) = 1$. The spaces of spherical harmonics are invariant under rotations. Let U denote an arbitrary rotation. The invariance is expressed by

$$Y_l^m(U\zeta) = \sum_k a_k^m(U)\, Y_l^k(\zeta), \qquad |\zeta| = 1 \tag{22}$$

The properties listed above imply the following lemma.

Lemma 3.1. Let l be an integer. Let $s_l^m(\zeta)$ denote a system of $2l+1$ spherical functions. Let these functions satisfy

$$s_l^m(U\zeta) = \sum_k a_k^m(U) s_l^k(\zeta) \tag{23}$$

[i.e., the same coefficients as in Eq. (22)]. Then the $s_l^m(\zeta)$ must be spherical harmonics corresponding to the weight l.

Another property is $\rho^l Y_l^m(\zeta) = P(\rho\zeta)$, where P is a homogeneous polynomial of degree l in the vector variable $\eta = \rho\zeta$. Conversely, any homogeneous polynomial of degree l can be uniquely expanded by

$$P(\eta) = \sum_{k=0}^{[l/2]} \rho^{2k} P_{l-2k}(\rho\zeta) \tag{24}$$

where P_{l-2k} are homogeneous of order $l - 2k$ and each of which can be expressed by

$$P_{l-2k}(\rho\zeta) = \rho^{l-2k} \sum_m \beta_m Y_l^m(\zeta) \tag{25}$$

The following general property will also be used. If a transformation T in an irreducible representation space commutes with all rotations, then it must be a multiple of the identity.

The arguments above will be used in the manipulation of (10). For the time being, take $K(\theta)$ to be an even kernel $[K(\theta) = K(\pi - \theta)$, or, equivalently, $K(\theta) = K(\sin\theta)]$.

Lemma 3.2. Define

$$S_l^m(\eta) = (1/2\pi) \int_{\zeta \perp \eta} Y_l^m(\zeta)\, d\varphi \tag{26}$$

where $|\zeta| = 1$ and φ is the polar angle in the plane perpendicular to η. Then $S_l^m(\eta)$ is a spherical harmonic.

Proof. Consider any orthogonal transformation U. Then

$$S_l^m(U\eta) = \frac{1}{2\pi} \int_{\zeta \perp U\eta} Y_l^m(\zeta)\, d\varphi = \frac{1}{2\pi} \int_{U^*\zeta \perp \eta} Y_l^m(\zeta)\, d\varphi = \frac{1}{2\pi} \int_{\xi \perp \eta} Y_l^m(U\xi)\, d\varphi \tag{27}$$

where $\xi = U^*\zeta$.

Observe that in the last passage we changed the circle on which integration was performed, in both cases using φ as the polar angle.

Substitute (22) on the right-hand side of (27) to get

$$S_l^m(U\eta) = (1/2\pi) \int_{\xi \perp \eta} \sum_k a_k^m(U) Y_l^k(\xi)\, d\varphi$$

$$= \sum_k a_k^m(U)(1/2\pi) \int_{\xi \perp \eta} Y_l^k(\xi)\, d\varphi = \sum_k a_k^m(U) S_l^k(\eta) \tag{28}$$

Thus, by Lemma 3.1, this lemma is established.

Lemma 3.3. The following holds:

$$S_l^m(\eta) = c(l) Y_l^m(\eta) \tag{29}$$

where

$$c(l) = 0, \qquad l \text{ odd} \tag{30}$$

$$c(l) = (-1)^{l/2} \frac{1 \cdot 3 \cdot 5 \cdots (l-1)}{2 \cdot 4 \cdot 6 \cdots l}, \qquad l \text{ even} \tag{31}$$

Proof. Denote, for spherical functions, $S(\zeta)$: $US(\zeta) = S(U\zeta)$. Define T: $Y_l^m \to S_l^m$. By Lemma 3.2 it is a transformation in an irreducible space of spherical harmonics. Thus

$$\begin{aligned} TUY_l^m(\zeta) &= TY_l^m(U\zeta) = T\sum_k a_k^m(U)Y_l^k(\zeta) \\ &= \sum_k a_k^m(U)TY_l^k(\zeta) = \sum_k a_k^m(U)S_l^k(\zeta) \\ &= S_l^m(U\zeta) = US_l^m(\zeta) = UTY_l^m(\zeta) \end{aligned} \tag{32}$$

T commutes with all rotations and consequently is a multiple of the identity. Thus (29) is established. As for the computation of $c(l)$, for that we can utilize a convenient choice of Y_l^m and η. Let us choose $\eta = (1, 0, 0)$ and $Y_l^m = P_l(\cos\theta)$, where P_l is the Legendre polynomial of order l, as exhibited in (21). Since $\eta = (1, 0, 0)$, the integration in (26) is performed on the circle $\theta = \pi/2$, i.e., $\cos\theta = 0$. In this case the right-hand side of (26) will be equal to the free term of $P_l(x)$, as exhibited on the right-hand side of (30) and (31). Since $Y_l^m(\eta) = P_l(1) = 1$, the lemma follows.

Lemma 3.4. Let $Q(\xi)$ be a homogeneous polynomial of order l. Then for ζ perpendicular to $\xi - \eta$ ($|\zeta| = 1$)

$$|\xi - \eta|^l(1/2\pi)\int Q(\zeta)\,d\varphi = P(\xi - \eta) \tag{33}$$

where P is a homogeneous polynomial of order l in the variable $\xi - \eta$. If l is odd, then $P(\xi - \eta) = 0$.

Proof. Consider, first, the case where

$$Q(\xi - \eta) = |\xi - \eta|^l Q(\zeta) = |\xi - \eta|^l Y_l^m(\zeta) \tag{34}$$

In this case, by (29) it follows that

$$\begin{aligned} |\xi - \eta|^l(1/2\pi)\int Q(\zeta)\,d\varphi &= |\xi - \eta|^l(1/2\pi)\int Y_l^m(\zeta)\,d\varphi \\ &= |\xi - \eta|^l c(l) Y_l^m(\zeta) = c(l)Q(\xi - \eta) \end{aligned} \tag{35}$$

As for the general case, use the expansions (24) and (25), which, together with (35), establish (33).

Observe also that if l is odd, then the expansions (24) and (25) contain only polynomials having odd degree. For each of these, by Lemma 3.3, the multipliers $c(l - 2k)$ vanish. Thus $P(\xi - \eta)$ vanishes identically.

The next step will be the integration $d\varphi$ of formula (10). In (10) expand the brackets [ ] to get

$$[\xi + \eta + (\cos\theta)(\xi - \eta) + (\sin\theta)|\xi - \eta|\zeta]^\beta$$
$$= \sum_{\alpha+\lambda+\mu=\beta} b_{\alpha,\lambda,\mu}(\xi + \eta)^\lambda(\cos\theta^{|\mu|})(\xi - \eta)^\mu(\sin\theta^{|\alpha|})|\xi - \eta)^\alpha\zeta^\alpha \quad (36)$$

where the summation extends over all nonnegative multiindices.

The right-hand side is now substituted in (10) to get

$$\omega_\beta = (1/2\pi)(1/2)^{|\beta|} \sum_{\alpha+\lambda+\mu=\beta} b_{\alpha,\lambda,\mu}$$
$$\times \iiint (\xi + \eta)^\lambda(\cos\theta^{|\mu|})|\xi - \eta|^{|\mu|}(\sin\theta^{|\alpha|})$$
$$\times \left[|\xi - \eta)^\alpha(1/2\pi)\int_{\zeta\perp\xi-\eta} \zeta^\alpha\, d\varphi\right]K(\theta, |\xi - \eta|)f(\xi)f(\eta)\, d\theta\, d\xi\, d\eta$$
$$- \int\left[\iint K(\theta, |\xi - \eta|)f(\eta)\, d\theta\, d\eta\right]\xi^\beta f(\xi)\, d\xi \quad (37)$$

By Lemma 3.4,

$$|\xi - \eta|^\alpha(1/2\pi)\int_{\zeta\perp\xi-\eta} \zeta^\alpha\, d\varphi = P_\alpha(\xi - \eta) \quad (38)$$

Substitute (38) in (37) and consider the following expressions in this formula:

$$(\xi + \eta)^\lambda|\xi - \eta|^{|\mu|}P_\alpha(\xi - \eta)\int \cos\theta^{|\mu|}\sin\theta^{|\alpha|}\, K(\theta, |\xi - \eta|)\, d\theta \quad (39)$$

Since K is a function of $\sin\theta$, it follows that for odd $|\mu|$ the integral in (39) vanishes. Thus the sum in (37) needs to be evaluated only for even $|\mu|$. For these, $|\xi - \eta|^{|\mu|}$ is a polynomial in $\xi - \eta$. Denote now

$$Q_\theta(\xi, \eta) = (1/2\pi)(1/2)^{|\beta|} \sum_{\substack{\alpha+\lambda+\mu=\beta \\ |\mu|\ \text{even}}} b_{\alpha,\lambda,\mu}(\xi + \eta)^\lambda|\xi - \eta|^{|\mu|}$$
$$\times P_\alpha(\xi - \eta)\cos\theta^{|\mu|}\sin\theta^{|\alpha|} \quad (40)$$

and again substitute in (37). The result is expressed by:

Lemma 3.5.

$$\omega_\beta = \iiint Q_\theta(\xi, \eta)K(\theta, |\xi - \eta|)f(\xi)f(\eta)\, d\xi\, d\eta\, d\theta$$
$$- \int\left[\iiint f(\eta)K(\theta, |\xi - \eta|)\, d\eta\, d\theta\right]\xi^\beta f(\xi)\, d\xi \quad (41)$$

where, for each θ, $Q_\theta(\xi, \eta)$ is a polynomial in the six variables ξ_i, η_i. It is homogeneous of order $|\beta|$.

Lemmas 3.2–3.4 will be used to compute the derivatives $d\psi_\beta/dt$. Even at this point it is clear that they offer a big advantage over a direct computation. [Compare to the verification of formula (18).]

4. GENERALIZED MAXWELLIAN LAW OF FORCE

This term refers to kernels that are independent of $\xi - \eta$. In this case a further reduction is possible, i.e., the integration $d\theta$ can be explicitly performed.

Lemma 4.1. For $K = K(\theta)$

$$\omega_\beta = \iint [Q_\beta(\xi, \eta) - c\xi^\beta] f(\xi) f(\eta)\, d\xi\, d\eta \tag{42}$$

where Q_β is a polynomial in the six variables ξ_i, η_i. The polynomial is homogeneous of order $|\beta|$, and c is the constant $\int K(\theta)\, d\theta$.

Proof. Consider the proof of Lemma 3.5. Denote, in this case,

$$Q_\beta(\xi, \eta) = (1/2\pi)(1/2)^{|\beta|} \sum_{\alpha+\lambda+\mu=\beta} b_{\alpha,\lambda,\mu}(\xi + \eta)^\lambda |\xi - \eta|^{|\mu|} P_\alpha(\xi - \eta)$$
$$\times \int \cos \theta^{|\mu|} \sin \theta^{|\alpha|} K(\theta)\, d\theta \tag{43}$$

Since, again, for odd $|\mu|$, the integral in (43) vanishes, it follows that $Q_\beta(\xi, \eta)$ is a polynomial in ξ and η.

Theorem 4.1. For $K = K(\theta)$

$$\omega_\beta = \sum_{|\alpha|+|\gamma|=|\beta|} a_{\alpha,\beta,\gamma}\psi_\alpha\psi_\gamma - c\psi_{(0,0,0)}\psi_\beta$$
$$= \psi_{(0,0,0)} \sum_{|\alpha|=|\beta|} a_{\alpha,\beta}\psi_\alpha + \sum_{0<|\alpha|<|\beta|} a_{\alpha,\beta,\gamma}\psi_\alpha\psi_\gamma - c\psi_{(0,0,0)}\psi_\beta \tag{44}$$

where the constants $a_{\alpha,\beta,\gamma}$ depend on $K(\theta)$.

Proof. Denote in (42)

$$Q_\beta(\xi, \eta) = \sum_{\alpha+\gamma=\beta} c_{\alpha,\gamma}\xi^\alpha\eta^\gamma$$

Since

$$\iint \xi^\alpha\eta^\gamma f(\xi)f(\eta)\, d\xi\, d\eta = \int \xi^\alpha f(\xi)\, d\xi \int \eta^\gamma f(\eta)\, d\eta = \psi_\alpha\psi_\gamma \tag{45}$$

the desired formula (44) is obtained.

Theorem 4.1 is due, in the special case of Maxwellian molecules, to Ikenberry.[11] The result given here is more general and the proof is considerably simpler than that of Ikenberry.

The reader has undoubtedly noticed that the expression $f(\xi)f(\eta)\,d\xi\,d\eta$ is carried unchanged throughout the computation [except for the relation (45)]. The lemmas that were established deal with polynomials defined on ξ space. Let us state now an isomorphism between moments and homogeneous polynomials as follows:

Basic Isomorphism. Associate with the moment ψ_β the homogeneous polynomial ξ^β. Then ω_β will be associated with $Q_\beta(\xi, \eta) - c\xi^\alpha$. The correspondence, expressed explicitly, is

$$\omega_\beta \leftrightarrow (1/2\pi)(1/2^{|\beta|}) \iint [\xi + \eta + (\cos\theta)(\xi - \eta) + (\sin\theta)|\xi - \eta|\zeta]^\beta K(\theta)\,d\theta\,d\varphi - \xi^\beta \int K(\theta)\,d\theta \tag{46}$$

where ζ is evenly distributed on a plane orthogonal to $\xi - \eta$.

In order to compute ω_β as a bilinear form in the moments [i.e., Eq. (44)], one has to compute $Q_\beta(\xi, \eta) - c\xi^\beta$, then substitute back moments ψ_α and ψ_γ instead of ξ^α and η^γ, respectively. In particular, the coefficients of the moments having highest order are obtained by considering the parts of $Q_\beta(\xi, \eta)$ that depend only on ξ or only on η. Denote these by $Q_\beta^{(1)}(\xi)$ and $Q_\beta^{(2)}(\eta)$, respectively. Equation (43) leads to

$$Q_\beta^{(1)}(\xi) = (1/2\pi)(1/2)^{|\beta|} \sum_{\alpha+\lambda+\mu=\beta} b_{\alpha,\lambda,\mu}\xi^\lambda|\xi|^{|\mu|} \times P_\alpha(\xi) \int \cos\theta^{|\mu|} \sin\theta^{|\alpha|} K(\theta)\,d\theta \tag{47}$$

$$Q_\beta^{(2)}(\eta) = (1/2\pi)(1/2)^{|\beta|} \sum_{\alpha+\lambda+\mu=\beta} b_{\alpha,\lambda,\mu}\eta^\lambda|-\eta|^{|\mu|} \times P_\alpha(-\eta) \int \cos\theta^{|\mu|} \sin\theta^{|\alpha|} K(\theta)\,d\theta \tag{48}$$

Since the polynomials P_α are homogeneous of order $|\alpha|$ and vanish for odd $|\alpha|$ and since $|-\eta| = |\eta|$, it follows that

$$Q_\beta^{(2)}(\eta) = Q_\beta^{(1)}(\eta) \tag{49}$$

The result is as follows: The computation of the coefficients of moments having highest order can be done by multiplying Eq. (47) by 2 and then substituting back the moments ψ_α for the monomials ξ^α.

The arguments above lead to the following characterization of the $a_{\alpha,\beta}$ in (44):

Lemma 4.2. Consider the space of homogeneous polynomials of order m. Choose as a basis the polynomials ξ^β: $|\beta| = m$. Then $a_{\alpha,\beta}$ will be the entries of the matrix representation of the transformation A_m in that space defined by

$$A_m\{\xi^\beta\} = \frac{1}{2\pi}\frac{1}{2^{m-1}}\iint [(1 + \cos\theta)\xi + (\sin\theta)|\xi|\zeta]^\beta K(\theta)\, d\theta\, d\varphi$$
$$= \sum_{|\alpha|=m} a_{\alpha,\beta}\xi^\alpha \tag{50}$$

where ζ is a unit vector perpendicular to ξ.

Proof. One has to show that the expression (50) is equal to twice expression (47). The middle part of (50) is evaluated in exactly the same way as the first term on the right of (46). Thus

$$\frac{1}{2\pi}\frac{1}{2^{m-1}}\iint [(1 + \cos\theta)\xi + (\sin\theta)\xi\zeta]^\beta K(\theta)\, d\theta\, d\varphi$$
$$= \frac{1}{2\pi}\frac{1}{2^{m-1}}\sum_{\alpha+\lambda+\mu=\beta} b_{\alpha,\gamma,\mu}\xi^\lambda|\xi|^{|\mu|}R_\alpha(\xi)$$
$$\times \int \cos\theta^{|\mu|}\sin\theta^{|\alpha|}\, K(\theta)\, d\theta \tag{51}$$

where

$$R_\alpha(\xi) = |\xi|^\alpha \int_{\zeta\perp\xi} \zeta^\alpha\, d\varphi \tag{52}$$

Recall that

$$P_\alpha(\xi - \eta) = |\xi - \eta|^\alpha \int_{\zeta\perp\xi-\eta} \zeta^\alpha\, d\varphi$$

It is clear that $P_\alpha(\xi) = R_\alpha(\xi)$. Recall also that $|\beta| = m$. Thus expression (51) and, consequently, (50) are equal to twice expression (47). QED

Lemma 4.3. Let $\xi = \rho\eta$, where $|\eta| = 1$. Then all the eigenfunctions of A_m can be expressed as

$$Q_{m,k,j}(\xi) = \rho^m Y^j_{m-2k}(\eta) \tag{53}$$

Proof. For any polynomial $Q(\xi)$

$$A_m Q(\xi) = \frac{1}{2\pi}\frac{1}{2^{m-1}}\iint Q[(1 + \cos\theta)\xi + (\sin\theta)|\xi|\zeta]K(\theta)\, d\theta\, d\varphi \tag{54}$$

where $\zeta \perp \xi$. Hence

$$A_m Q_{m,k,j}(\xi) = \frac{1}{2\pi}\frac{1}{2^{m-1}}\iint Q_{m,k,j}[(1+\cos\theta)|\xi|\eta + (\sin\theta)|\xi|\zeta]K(\theta)\,d\theta\,d\varphi$$
$$= \frac{1}{2\pi}\frac{1}{2^{m-1}}\rho^m \iint Q_{m,k,j}[(1+\cos\theta)\eta + (\sin\theta)\zeta]K(\theta)\,d\theta\,d\varphi$$

Since $\zeta \perp \eta$, it follows that

$$|(1+\cos\theta)\eta + (\sin\theta)\zeta| = (2+2\cos\theta)^{1/2}$$

Therefore, by homogeneity of $Q_{m,k,j}$,

$$A_m Q_{m,k,j}(\xi) = \frac{1}{2\pi}\frac{1}{2^{m-1}}\rho^m \iint (2+2\cos\theta)^{m/2} Y^j_{m-2k}(\beta)K(\theta)\,d\theta\,d\varphi \quad (55)$$

where β ($|\beta| = 1$) is a vector parallel to $(1+\cos\theta)\eta + (\sin\theta)\zeta$. (This β is not an exponent.)

Apply now the "representation of the orthogonal group" arguments, this time to polynomials defined in the whole space:

$$A_m Q_{m,k,j}(U\xi)$$
$$= \frac{1}{2\pi}\frac{1}{2^{m-1}}\iint Q_{m,k,j}[(1+\cos\theta)U\xi + (\sin\theta)|\xi|\zeta^*]K(\theta)\,d\theta\,d\varphi$$
$$= \frac{1}{2\pi}\frac{1}{2^{m-1}}\iint Q_{m,k,j}[(1+\cos\theta)U\xi + (\sin\theta)|\xi|U\zeta]K(\theta)\,d\theta\,d\varphi$$

where $\zeta^* \perp U\xi$, $\zeta \perp \xi$ (consequently $\zeta \perp \eta$). Note also here the change in the domain of the integration $d\varphi$ [cf. the remark following (27)].

Since $U\zeta \perp U\eta$ it follows that

$$|(1+\cos\theta)U\eta + (\sin\theta)U\zeta| = (2+2\cos\theta)^{1/2}$$

Therefore

$$A_m Q_{m,k,j}(U\xi) = \frac{1}{2\pi}\frac{1}{2^{m-1}}\rho^m \iint (2+2\cos\theta)^{m/2} Y^j_{m-2k}(U\beta)K(\theta)\,d\theta\,d\varphi$$

where β was defined in (55).

The last expression can be rewritten as [cf. (22)]

$$\frac{1}{2\pi}\frac{1}{2^{m-1}}\rho^m \iint (2+2\cos\theta)^{m/2}\Big[\sum_i a_i^j(U)\, Y^i_{m-2k}(\beta)\Big]K(\theta)\,d\theta\,d\varphi$$

The $a_i^j(U)$ can be taken out of the integral to yield

$$A_m Q_{m,k,j}(U\xi) = \sum_i a_i^j(U) A_m Q_{m,k,i}(\xi)$$

For fixed m and k the index j takes the values $-(m-2k),\ldots,0,\ldots,m-2k$ [cf. (53)]. The span of $\{Q_{m,k,j}\}$ is an irreducible space, this time composed of homogeneous polynomials. These homogeneous polynomials must coincide, on the unit sphere, with spherical harmonics and A_m must be, on each irreducible space, a multiple of the identity. Thus the $Q_{m,k,j}(\xi)$ are indeed eigenfunctions of A_m. The eigenfunctions $Q_{m,k,j}$ form a basis for the space of homogeneous polynomials of order m. Therefore these are all the eigenfunctions.

Lemma 4.4. The corresponding eigenvalues (of A_m) are

$$\lambda_{m,k,j} = \lambda_{m,k} = (1/2^{m-1}) \int (2 + 2\cos\theta)^{m/2} P_{m-2k}(\cos\tfrac{1}{2}\theta) K(\theta)\, d\theta \tag{56}$$

Proof. It is already known that the eigenvalues are independent of j. Let us choose a convenient spherical harmonic, i.e., $Q_{m,k,0}$:

$$Q_{m,k,0}(\xi) = \rho^m Y^0_{m-2k}(\eta) = \rho^m P_{m-2k}(\cos t), \qquad \cos t = (e_3, \eta) \tag{57}$$

Take also a convenient ξ: $\xi = e_3 = (0, 0, 1)$. Thus $Q_{m,k,0}(e_3) = 1$, and by (54),

$$A_m Q_{m,k,0}(e_3) = \frac{1}{2\pi}\frac{1}{2^{m-1}} \int Q_{m,k,0}[(1 + \cos\theta)e_3 + (\sin\theta)\zeta] K(\theta)\, d\theta\, d\varphi \tag{58}$$

where $\zeta \perp e_3$.

Substituting (53) into (54) and noting that the magnitude of $(1 + \cos\theta)e_3 + (\sin\theta)\zeta$ is $(2 + 2\cos\theta)^{1/2}$, one gets

$$A_m Q_{m,k,0}(e_3) = (1/2^{m-1}) \int (2 + 2\cos\theta)^{m/2} Y^0_{m-2k}(\beta) K(\theta)\, d\theta \tag{59}$$

where $|\beta| = 1$,

$$\beta = (2 + 2\cos\theta)^{-1/2}[(1 + \cos\theta)e_3 + (\sin\theta)\zeta]$$

Since $\zeta \perp e_3$ it follows that

$$(\beta, e_3) = (1 + \cos\theta)(2 + 2\cos\theta)^{-1/2} = [\tfrac{1}{2}(1 + \cos\theta)]^{1/2} = \cos\tfrac{1}{2}\theta$$

We substitute now in (59) to get

$$A_m Q_{m,k,0}(e_3) = (1/2^{m-1}) \int (2 + 2\cos\theta)^{m/2} P_{m-2k}(\cos\tfrac{1}{2}\theta) K(\theta)\, d\theta$$

which is the desired result (56).

Lemma 4.5. Let $c = \int K(\theta)\, d\theta$. Suppose that $K(\theta)$ is not supported just by the points $\theta = 0$ and $\theta = \pi$. Then

$$\lambda_{1,0} = \lambda_{2,1} = c \tag{60}$$

$$\lambda_{2,0} < c \tag{61}$$

$$\lambda_{m,k} < c, \qquad m \geqslant 3 \tag{62}$$

and $\lambda_{m,k} \to 0$ as $m \to \infty$.

Remark. If $K(\theta)$ is supported only by 0 and π, it means that two colliding molecules do not change their velocity.

Proof of the Lemma. For $\lambda_{1,0}$ substitute $P_1(\cos \frac{1}{2}\theta) = \cos \frac{1}{2}\theta$, for $\lambda_{2,1}$ substitute $P_0 = 1$, and integrate. The computation of $\lambda_{2,0}$ is instructive and will be needed later. So

$$\begin{aligned}\lambda_{2,0} &= (1/2)\int (1 + \cos\theta)(3\cos^2 \tfrac{1}{2}\theta - 1)K(\theta)\, d\theta \\ &= (1/4)c + (3/4)c_2\end{aligned} \tag{63}$$

where

$$c_2 = \int \cos^2 \theta\, K(\theta)\, d\theta \tag{64}$$

Since $c_2 < c$ the result follows. As for $m \geqslant 3$, use the fact that $K(\theta) = K(\pi - \theta)$ and the simple inequality

$$(1 + \cos\theta)^{m/2} + [1 + \cos(\pi - \theta)]^{m/2} = (1 + \cos\theta)^{m/2} + (1 - \cos\theta)^{m/2} < 2^{m/2} \tag{65}$$

Since for $-1 \leqslant x \leqslant 1$, $|P_j(x)| \leqslant 1$, substitute 1 instead of $P_{m-2k}(\cos \frac{1}{2}\theta)$ and estimate the quantities that follow:

$$\begin{aligned}&\int_0^\pi (1 + \cos\theta)^{m/2} P_{m-2k}(\cos \tfrac{1}{2}\theta) K(\theta)\, d\theta \\ &\qquad \leqslant \int_0^\pi (1 + \cos\theta)^{m/2} K(\theta)\, d\theta \\ &\qquad = \int_0^{\pi/2} [(1 + \cos\theta)^{m/2} + (1 - \cos\theta)^{m/2}] K(\theta)\, d\theta \\ &\qquad < \int_0^{\pi/2} 2^{m/2} K(\theta)\, d\theta = 2^{m/2-1} c\end{aligned}$$

Substitution of the last inequality in (56) yields (62).

As shown in the last inequality,

$$|\lambda_{m,k}| \leqslant 2\int_0^{\pi/2} \left(\frac{1 + \cos\theta}{2}\right)^{m/2} K(\theta)\, d\theta$$

As m tends to ∞ the integrand tends to zero uniformly on any interval $[\epsilon, \pi]$. Thus the last assertion of the lemma is established.

Sometimes it is better to use an equivalent expression for $\lambda_{m,k}$:

$$\lambda_{m,k} = 2\int (\cos\tfrac{1}{2}\theta)^m P_{m-2k}(\cos\tfrac{1}{2}\theta) K(\theta)\, d\theta, \qquad m \geqslant 2 \tag{66}$$

It is interesting to note that similar formulas hold for the linearized Boltzmann equation (cf. Ref. 4).

Remark. The basic isomorphism is a correspondence between polynomials and moments. Let us denote the moments corresponding to $Q_{m,k,j}(\xi)$ by $\phi_{m,k,j}$. These are called "spherical moments" (cf. Ref. 10).

5. THE SPATIALLY HOMOGENEOUS MAXWELL–BOLTZMANN EQUATION AND ASYMPTOTIC EXPANSIONS

If $f(x, \xi, t)$ is independent of x, then Eq. (1) reduces to

$$\partial f/\partial t = J(f, f) \tag{67}$$

Let us multiply it by ξ^β and integrate $d\xi$. Then, by (9),

$$d\psi_\beta/dt = \omega_\beta \tag{68}$$

The ω_β were studied in the last section. In particular, when $K = K(\theta)$, ω_β is polynomial in the moments ψ_α: $|\alpha| \leqslant |\beta|$. Let us recall that the $Q_{m,k,j}(\xi)$ form a basis for the space of homogeneous polynomials of order m. Thus, by the basic isomorphism, the spherical moments $\phi_{m,k,j}$ form a basis for the space of moments of order m. Theorem 4.1 together with Lemma 4.4 may be restated as follows:

Lemma 5.1. For $K = K(\theta)$

$$d\phi_{m,k,j}/dt = \psi_{(0,0,0)}[\lambda_{m,k} - c]\phi_{m,k,j} + P_{m,k,j}(\phi_{l,r,s}) \tag{69}$$

where $l < m$ and $P_{m,k,j}$ is a quadratic polynomial.

Remark. $P_{m,k,j}$ may depend on j. Its exact form may be calculated explicitly or by the use of products of representations. Both ways are very complicated in the general case. I could not find a simple reduction as in the highest order case.

Theorem 5.1. For generalized Maxwellian molecules $\lim_{t\to\infty} \psi_\beta(t)$ exists for all β and

$$\lim_{t\to\infty} \psi_\beta(t) = F_\beta(\psi_{(0,0,0)}, \psi_{(1,0,0)}, \psi_{(0,1,0)}, \psi_{(0,0,1)}, E) \tag{70}$$

The F_β are polynomials in their arguments.

Remark. The arguments in F_β are the familiar density, momentum, and energy. It is clear that they are independent of t. The constancy of the basic five moments was derived as an exercise in Section 2. There it was shown that the corresponding ω_β vanish. Thus, by (68) the respective ψ_β are constant.

Proof of Theorem 5.1. By induction on m. For $m = 0, 1$ the $\phi_{m,k,j}$ are the invariants. For $m = 2$ the energy E is $\phi_{2,1}$, again an invariant. For all the other $\phi_{m,k,j}$ ($m \geqslant 3$ and five spherical moments for $m = 2$) it follows, by Lemma 4.5, that

$$\mu_{m,k} = \psi_{(0,0,0)}[\lambda_{m,k} - c] < 0 \tag{71}$$

Equation (69) can be integrated to yield

$$\phi_{m,k,j}(t) = [\exp(\mu_{m,k}t)]\left\{\int_0^t [\exp(-\mu_{m,k}\tau)]P_{m,k,j}(\phi_{l,r,s}(\tau))\,d\tau + \phi_{m,k,j}(0)\right\} \tag{72}$$

Since $l < m$, $\phi_{l,k,j}(t)$ converges to a limit as $t\to\infty$ by the induction hypothesis. Therefore $P_{m,k,j}(\phi_{l,k,j}(t))$ also converges. In the limit, by the induction hypothesis,

$$\lim_{t\to\infty} \phi_{l,k,j}(t) = G_{l,k,j}(\psi_{(0,0,0)}, \psi_{(1,0,0)}, \psi_{(0,1,0)}, \psi_{(1,0,0)}, E)$$

(the $\phi_{l,k,j}$ can be expressed by the ψ_β and vice versa!). $P_{m,k,j}$ is a polynomial in the moments; therefore $P_{m,k,j}(G_{l,k,j}(\cdot))$ is also a polynomial. Now take the limit in (72) to get

$$\lim_{t\to\infty} \phi_{m,k,j}(t) = (\mu_{m,k})^{-1}P_{m,k,j}(G_{l,k,j}(\cdot))$$

which is the required form for order m.

Once the theorem is established, it is relatively simple to compute the F_β. Use a particular distribution, i.e., the celebrated Maxwell–Boltzmann distribution

$$g(\xi) = b\exp(-a|\xi - \bar{\xi}|^2)$$

This distribution is invariant under collisions. Moreover, it is determined by the five parameters b, a, $\bar{\xi}$. It follows that $\psi_{(0,0,0)} = (1/2)(2\pi)^{3/2}a^{-3/2}b$, $\psi_{(1,0,0)} = \bar{\xi}_1\psi_{(0,0,0)}$, and $E = (3/4)a^{-1}\psi_{(0,0,0)}$. So, for this distribution one can compute the moments explicitly, then express them in terms of the invariants. Theorem 5.1 is known for two special cases: elastic spheres[2] and

Maxwellian molecules.[11] The proof above is considerably simpler than previously constructed proofs.

The problem of the rate of decay is considered next. For that problem (70) should be replaced by asymptotic estimates. Let us consider a more general problem. Suppose that an estimate of a certain moment in terms of lower order moments is desired. What estimates can be established? Do we get better asymptotic estimates by allowing the order of the estimators to be higher? It will be shown that the answer is yes.

Our starting point is formula (69) with the substitution of $\mu_{m,k}$ [as defined in (71)]

$$d\phi_{m,k,j}/dt = \mu_{m,k}\phi_{m,k,j} + P_{m,k,j}(\phi_{l,r,s}) \tag{73}$$

Let us state the following elementary lemma.

Lemma 5.2. Suppose that $\phi_{m,k,j}(t)$ satisfies the differential equation

$$d\phi_{m,k,j}/dt = \mu_{m,k}\phi_{m,k,j} + \sum_i c_i \exp(\nu_i t) \tag{74}$$

where $\nu_i \neq \mu_{m,k}$. Then

$$\phi_{m,k,j}(t) = \left[\phi_{m,k,j}(0) - \sum_i c_i(\nu_i - \mu_{m,k})^{-1}\right] \exp(\mu_{m,k} t) + \sum_i c_i(-\mu_{m,k} + \nu_i)^{-1} \exp(\nu_i t) \tag{75}$$

Proof. Compute.

In (73) a spherical moment of order m, i.e., $\phi_{m,k,j}$, is related to spherical moments of lower order. In the general case not all the spherical moments of order less than m are arguments in the polynomial $P_{m,k,j}(\cdot)$. The following definition is a natural consequence.

Definition. $\phi_{m,k,j}$ is a direct successor of $\phi_{l,r,s}$ if in formula (73) $\phi_{l,r,s}$ is one of the arguments in the polynomial $P_{m,k,j}(\cdot)$. $\phi_{m,k,j}$ succeeds $\phi_{l,r,s}$ if there is a chain of direct successors leading from $\phi_{l,r,s}$ to $\phi_{m,k,j}$.

It turns out that the computations are simpler if $\phi_{m,k,j}(t)$ can be expressed as a sum of exponentials. Observe that if $\phi_{l,r,s}(t)$ can be expressed as a sum of exponentials, then $P_{m,k,j}(\phi_{l,r,s}(t))$ is also a sum of exponentials.

Thus the case where the $\phi_{m,k,j}$ can be expressed as a sum of exponentials will be treated first. This will be done via a technical condition (Condition I) defined recursively as follows: $\phi_{m,k,j}(t)$ satisfies Condition I if:

(i) All $\phi_{l,r,s}(t)$ preceding $\phi_{m,k,j}(t)$ can be expressed as a sum of exponentials. Furthermore, the $\phi_{l,r,s}(t)$ should also satisfy Condition I.
(ii) None of the exponents appearing in $P_{m,k,j}(\phi_{l,r,s})$ is equal to $\mu_{m,k}t$.

The basic invariants certainly satisfy Condition I.

Lemma 5.3. If $\phi_{m,k,j}$ satisfies Condition I, then there exists a polynomial $R_{m,k,j}$ in the variables $\phi_{l,r,s}$, $l < m$, for which

$$\phi_{m,k,j}(t) - R_{m,k,j}(\phi_{l,r,s}(t)) = d_{m,k,j} \exp(\mu_{m,k} t) \tag{76}$$

Proof. By "tree induction." Suppose that all the spherical moments preceding a certain spherical moment satisfy

$$\begin{aligned} \phi_{m,k,j}(t) &= d_{m,k,j} \exp(\mu_{m,k} t) + \sum_i c_{i,m,k,j} \exp(\nu_{i,m,k,j} t) \\ &= d_{m,k,j} \exp(\mu_{m,k} t) + R_{m,k,j}(\phi_{l,r,s}(t)) \end{aligned} \tag{77}$$

where

$$\nu_{i,m,k,j} = \sum_{l<m} n_{l,i} \mu_{l,r} \tag{78}$$

so that $n_{l,i} \neq 0$ if, for some r and s, $d_{l,r,s} \neq 0$. In order to "climb the tree," proceed as follows: Denote the $\phi_{m,k,j}$ for which, in (70), $d_{m,k,j} \neq 0$ by "spherical moments of the first kind." If, accordingly, $d_{m,k,j} = 0$, the corresponding $\phi_{m,k,j}$ will be of the second kind.

Add now two more statements to the induction hypothesis.

Each spherical moment of the second kind is a polynomial in spherical moments of the first kind.

If, for some j, $\phi_{m,k,j}$ is of the first kind, then the function $\exp(\mu_{m,k} t)$ can be expressed as a polynomial in $\phi_{m,k,j}$ and spherical moments of the first kind: $\phi_{l,r,s}$ where $l < m$.

Start now the induction cycle with (69). $P_{m,k,j}(\phi_{l,r,s})$ is a polynomial in spherical moments preceding $\phi_{m,k,j}$. By the induction hypothesis these can be expressed in terms of spherical moments of the first kind.

Each of these, by (77), is a sum of exponentials for which (78) is satisfied. Upon substitution in (77) one gets

$$d\phi_{m,k,j}/dt = \mu_{m,k} \phi_{m,k,j} + \sum_i c_{i,m,k,j} \exp(\nu_{i,m,k,j}) \tag{79}$$

Hence, by Lemma 5.3, the left-hand side of (77) is established. Note that $\nu_{i,m,k,j}$ is of the form (78), where on the right-hand side the index l, r (attached to all the $\phi_{l,r,s}$) precedes m, k. Therefore, by the induction hypothesis, each $\exp(\nu_{i,m,k,j} t)$ is expressed by a polynomial in the spherical moments of the first kind. Thus the right-hand side of (77) is also established.

Now, if for some k, j, $d_{m,k,j} = 0$, then $\phi_{m,k,j}$ is declared to be of the second kind and by the left-hand side of (77) it can be expressed as a polynomial in the spherical moments of the first kind.

If, for some k, j, $d_{m,k,j} \neq 0$, then $\phi_{m,k,j}$ is found to be of the first kind.

Then read (77) as

$$\exp(\mu_{m,k}t) = d_{m,k,j}^{-1}[\phi_{m,k,j}(t) - R_{m,k,j}(\phi_{l,r,s}(t))] \tag{80}$$

The right-hand side is, indeed, a polynomial in the spherical moments of the first kind. Thus the induction cycle is complete.

Lemma 5.3 is based on a correspondence between moments and exponentials, as expressed in (77) and (80). If Condition I is not satisfied, i.e., in (78) one of the exponents ν_i is equal to $\mu_{m,k}$, then Eq. (75) has to involve the term $t\exp(\mu_{m,k}t)$. Thus, by a reasoning similar to that of the proof of Lemma 5.3 the following is established.

Lemma 5.4. If Condition I is satisfied for all $\phi_{l,r,s}$ preceding $\phi_{m,k,j}$ but is not satisfied for $\phi_{m,k,j}$, then

$$\phi_{m,k,j}(t) - R_{m,k,j}(\phi_{l,r,s}(t)) = d_{m,k,j}\exp(\mu_{m,k}t) + \bar{d}_{m,k,j}t\exp(\mu_{m,k}t) \tag{81}$$

Thus we no longer retain the correspondence between moments and exponentials. But, in this case, the fact that Condition I is not satisfied means that $\exp(\mu_{m,k}t)$ already is expressed as a polynomial in spherical moments preceding $\phi_{m,k,j}$. Therefore,

$$t\exp(\mu_{m,k}t) = \bar{d}_{m,k,j}^{-1}[\phi_{m,k,j}(t) - \bar{R}_{m,k,j}(\phi_{l,r,s}(t))] \tag{82}$$

Passing now to the general case, we set up a correspondence between spherical moments and powers times exponentials.

Theorem 5.2. For any spherical moment $\phi_{m,k,j}$ there exists a polynomial $R_{m,k,j}$ in spherical moments of lower order so that

$$\phi_{m,k,j}(t) - R_{m,k,j}(\phi_{l,r,s}(t)) = \exp(\mu_{m,k}t)\sum_i d_{m,k,j,i}t^i \tag{83}$$

For the proof one needs the following:

Lemma 5.5. Suppose that $\phi_{m,k,j}(t)$ satisfies the differential equation

$$d\phi_{m,k,j}(t)/dt = \mu_{m,k}\phi_{m,k,j} + \sum_i \sum_{n=1}^{N_i} c_{i,n}[\exp(\nu_i t)]t^n \tag{84}$$

Then, if none of the ν_i is equal to $\mu_{m,k}$,

$$\phi_{m,k,j}(t) = d_{m,k,j}\exp(\mu_{m,k}t) + \sum_i \sum_{n=1}^{N_i} d_{i,n}[\exp(\nu_i t)]t^n \tag{85}$$

If one of the ν_i, say ν_j, is equal to $\mu_{m,k}$, then

$$\phi_{m,k,j}(t) = d_{m,k,j}\exp(\mu_{m,k}t) + \sum_i \sum_{n=1}^{N_i} d_{i,n}[\exp(\nu_i t)]t^n + d_{j,N_j+1}[\exp(\mu_{m,k}t)]t^{N_j+1} \tag{86}$$

Proof. Obvious.

Proof of Theorem 5.2. Again by a reasoning similar to that of Lemma 5.1, prove inductively and start with the expression $P_{m,k,j}(\phi_{l,r,s})$. Each of the $\phi_{l,r,s}$ can be expressed in terms of powers times exponentials. Hence $P_{m,k,j}(\phi_{l,r,s})$ also is expressible in terms of powers times exponentials. Thus, in analogy to (79) we get

$$d\phi_{m,k,j}/dt = \mu_{m,k}\phi_{m,k,j} + \sum_{i}\sum_{n=1}^{N_i} c_{i,m,k,j,n}[\exp(r_{i,m,k,j}t)]t^n \tag{87}$$

Each of the terms in the double sum on the right-hand side of (87) is expressible in terms of moments.

When integrating (87), either (86) or (85) is obtained if, accordingly, one of the $\nu_{i,m,k,j}$ is equal to $\mu_{m,k}$ or is not.

If (85) holds, then $\exp(\mu_{m,k}t)$ can be expressed as in (80). If (86) holds, $t^{N_j+1}\exp(\mu_{m,k}t)$ can be expressed by moments, taking into account that, for $n \leqslant N_j$, $t^n \exp(\mu_{m,k}t)$ had already been expressed in terms of moments preceding $\phi_{m,k,j}$. In either case (83) is established. The expression for $t^{N_j+1}\exp(\mu_{m,k}t)$ is used in the subsequent induction step. Thus the theorem is established.

Corollary to Theorem 5.2. Denote $\lim_{t\to\infty}\phi_{m,k,j}(t)$ by $\phi^*_{m,k,j}$. Then

$$\phi^*_{m,k,j} = R_{m,k,j}(\phi^*_{l,r,s}) \tag{88}$$

Proof. Take the limit in (83).

6. THE THERMODYNAMIC EQUATIONS

Let us consider the Maxwell–Boltzmann equation (1). The approximation of this equation by thermodynamic equations constitutes a basic problem in kinetic theory [a thermohydrodynamic system is a system of equations involving a finite number of the ξ-moments of the distribution $f(x, \xi, t)$]. Rigorous results so far are "meager"[14] and the reasoning is almost exclusively heuristic and formal.

In this section we suggest a method of constructing a hierarchy of thermohydrodynamic systems. It is based on the theorems obtained for generalized Maxwellian molecules. A good part of the reasoning is still heuristic.

For that purpose multiply Eq. (1) by ξ^β and integrate $d\xi$. One gets

$$\begin{aligned}\frac{d\psi_\beta}{dt}(x,t) &+ \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\psi_{\beta+e_i}(x,t)\\ &= \psi_{(0,0,0)}(x,t)\sum_{|\alpha|=|\beta|} a_{\alpha,\beta}\psi_\alpha(x,t)\\ &\quad + \sum_{|\alpha|+|\gamma|=|\beta|} a_{\alpha,\beta,\gamma}\psi_\alpha(x,t)\psi_\gamma(x,t) - c\psi_{(0,0,0)}(x,t)\psi_\beta(x,t)\end{aligned} \tag{89}$$

where $\beta + e_1 = (\beta_1 + 1, \beta_2, \beta_3)$. This formula holds only for generalized Maxwellian molecules and follows from Eq. (44).

Since $|\beta| = m$ implies $|p + e_i| = m + 1$, the system (89) cannot be decomposed into finite subsystems.

Let us exhibit an equivalent form to (89) which is invariant under rotations. The variables in this system will naturally be the spherical moments.

Equation (69) exhibits the system in the spatially homogeneous case. When the x dependence is taken into account, the full system turns out to be

$$\frac{d\phi_{m,k,j}(x, t)}{dt} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \phi^*_{m,k,j,i}(x, t)$$

$$= \psi_{(0,0,0)}(\lambda_{m,k} - c)\phi_{m,k,j}(x, t) + P_{m,k,j}(\phi_{l,r,s}(x, t)) \tag{90}$$

The $\phi^*_{m,k,j,i}$ are combinations of moments having order $m + 1$. These combinations are obtained by expressing the spherical moments as a combination of the moments ψ_β, then replacing ψ_β by $\psi_{\beta+e_i}$, and finally transforming back to spherical moments. Some explicit computations are carried out in Section 7.

The problem of the construction of approximating (or limiting) thermohydrodynamic systems may be split into two parts.

Part A: How to choose a finite set of spherical moments to be the variables in a thermohydrodynamic system.

Part B: How to "close" the system. The term "closure" refers to the substitution in (90) of any moment not chosen to be a variable by a combination of the variables of the system.

In the case of generalized Maxwellian molecules, the "collision term" [i.e., the right hand side of (90)] does not pose a closure problem. One has, though, to impose the following consistency condition. If $\phi_{m,k,j}$ is chosen, then all the $\phi_{l,r,s}$, i.e. the variables in the polynomial $P_{m,k,j}$, have also to be chosen.

Let us turn to Part A. It is well established that the fundamental system consists of the basic five invariants: density, momentum, and energy. These correspond to the spherical moments $\phi_{0,0,1}$, $\phi_{1,0,j}$, and $\phi_{2,1,1}$, respectively. (In the sequel $\phi_{0,0}$ and $\phi_{2,1}$ will be used in place of $\phi_{0,0,1}$ and $\phi_{2,1,1}$, respectively. In general, if the j index will be only the integer one, it will be suppressed.)

Let us review a possible line of reasoning that suggests the choice of the fundamental system. Consider the spatially homogeneous equation $df/dt = (1/\epsilon)J(f,f)$, where ϵ is some small parameter. Change the time variable $\tau = t/\epsilon$. Thus $df/d\tau = J(f,f)$. It follows that keeping t fixed and letting ϵ tend to zero is equivalent to letting τ tend to infinity. Hence, in case the collision term is large, after a finite time t all the moments will be close to their limiting values, as expressed in Theorem 5.1.

Consider now the inhomogeneous equation. Choose some point x_0 and compare the spatially inhomogeneous equation around x_0 to the spatially homogeneous equation at x_0. For the spatially homogeneous case, if ϵ is small, after a short time interval the ξ-moments of the distribution $f(x, \xi, t)$ will be close to their limiting expressions. For the spatially inhomogeneous equation, if ϵ is small indeed, the streaming term will not change very much the relations between the ξ-moments (heuristic reasoning!). Thus the basic five invariants are chosen also in the inhomogeneous case. For further discussion the reader is referred to Truesdell.[11,14]

Suppose, now, that a finer approximation is desired. For that we suggest the following reasoning. Consider, not just the limit, but also the asymptotic behavior of the moments, as expressed by Theorem 5.2. Specify $\exp(\mu t)$, a rate of decay, and consider all moments for which $\mu_{m,k} \geqslant \mu$. These will be the variables in a thermohydrodynamic system.

Since $\mu_{m,k} = \psi_{(0,0,0)}(\lambda_{m,k} - c)$, one may also base the decision upon the $\lambda_{m,k}$. Let us compute these for the moments having lowest order. It is already known that $\lambda_{1,0} = \lambda_{2,1} = c_0 = \int K(\theta)\, d\theta$ and that $\lambda_{2,0} = \frac{1}{4}c_0 + \frac{3}{4}c_2$, where $c_2 = \int \cos^2 \theta\, K(\theta)\, d\theta$. Now

$$\lambda_{3,1} = 2\int (\cos^3 \tfrac{1}{2}\theta)(\cos \tfrac{1}{2}\theta)K(\theta)\, d\theta$$

$$= \tfrac{1}{2}\int (1 + \cos\theta)^2 K(\theta)\, d\theta = \tfrac{1}{2}(c_0 + c_2) \qquad (91)$$

$$\lambda_{3,0} = 2\int (\cos^3 \tfrac{1}{2}\theta)P_3(\cos \tfrac{1}{2}\theta)K(\theta)\, d\theta = -\tfrac{1}{8}c_0 + \tfrac{9}{8}c_2 \qquad (92)$$

$$\lambda_{4,2} = 2\int (\cos^4 \tfrac{1}{2}\theta)K(\theta)\, d\theta = \tfrac{1}{2}(c_0 + c_2) \qquad (93)$$

$$\lambda_{4,1} = 2\int (\cos^4 \tfrac{1}{2}\theta)P_2(\cos \tfrac{1}{2}\theta)K(\theta)\, d\theta = \tfrac{1}{8}c_0 + \tfrac{7}{8}c_2 \qquad (94)$$

$$\lambda_{4,0} = 2\int (\cos^4 \tfrac{1}{2}\theta)P_4(\cos \tfrac{1}{2}\theta)K(\theta)\, d\theta$$

$$= (35/64)c_4 + (21/32)c_2 - (13/64)c_0 \qquad (95)$$

[where $c_4 = \int (\cos^4 \theta)K(\theta)\, d\theta$]

$$\lambda_{5,2} = 2\int (\cos^5 \tfrac{1}{2}\theta)(\cos \tfrac{1}{2}\theta)K(\theta)\, d\theta = \tfrac{1}{4}c_0 + \tfrac{3}{4}c_2 \qquad (96)$$

$$\lambda_{5,1} = 2\int (\cos^5 \tfrac{1}{2}\theta)P_3(\cos \tfrac{1}{2}\theta)K(\theta)\, d\theta$$

$$= \tfrac{5}{16}c_4 + \tfrac{3}{4}c_2 - \tfrac{1}{16}c_0 \qquad (97)$$

Thus, if the choice of moments is based upon the asymptotic behavior, the second-order approximating system will consist of the basic five invariants, the three moments $\phi_{3,1,j}$ (the "heating-flux vector"), and the spherical moment $\phi_{4,2}$. In total, the system will consist of nine equations in nine variables.

Observe that, except for the energy, no second-order moment is included in the second-order approximation. The full set of second-order moments ("the stress tensor") does appear in the third-order approximation. Since $\lambda_{5,2} = \lambda_{2,0}$, it follows that the three spherical moments $\phi_{5,2,j}$ have also to be included. Thus the third-order system will consist of 17 equations in 17 variables.

Let us turn to Part B. What expressions should one substitute for the $\phi^*_{m,k,j,i}$ (or for the $\psi_{\beta+e_i}$)? The answer is furnished, again, by Theorem 5.2 and is as follows.

If some moment $\phi_{m,k,j}$ is not chosen to be an independent variable, substitute for it the expression $P_{m,k,j}(\phi_{l,r,s})$, which is the corresponding approximation in (69). If, in turn, one of the $\phi_{l,r,s}$ is not an independent variable, substitute for it the respective approximation, and so on. The procedure just described terminates in at most m steps.

The asymptotic formulas for the spherical moments of order two and three that appear in the formulation of the second and third approximations will be exhibited below.

7. EXPLICIT COMPUTATION OF SOME ASYMPTOTIC FORMULAS AND EQUATIONS

The computation of the $a_{\alpha,\beta}$ was carried out by passage to an equivalent formulation, i.e., the problem was reformulated in terms of transformations acting in the space of homogeneous polynomials. Now we consider lower order terms. The equivalent problem will be set up as a transformation from H_m, the space of homogeneous polynomials of degree m in $\xi = (\xi_1, \xi_2, \xi_3)$ to V_m, the space of homogeneous polynomials in ξ and η (also of total degree m). This transformation commutes with rotations, so one is tempted to break V_m into a direct sum of irreducible subspaces. Unfortunately, many of the irreducible spaces in V_m form equivalence classes with respect to isomorphism. One can still try to work with known formulas for products of representations. It seems, however, that in this case a direct explicit computation is the fastest way. Some of the considerations for products of representations help to check the results obtained.

The basic computation will be that of the transformation T defined by

$$Q(\xi) \to (1/2\pi)(1/2^m) \int Q[\tfrac{1}{2}(\xi + \eta) + \tfrac{1}{2}(\cos\theta)(\xi - \eta) + \tfrac{1}{2}(\sin\theta)\zeta|\xi - \eta|]K(\theta)\,d\theta\,d\varphi - \int K(\theta)\,d\theta Q(\xi) \tag{98}$$

where $Q(\xi)$ is a homogeneous polynomial. As an illustration, let us carry out the computation

$$\begin{aligned}\xi_1\xi_2 \to (1/2\pi)\tfrac{1}{4} \iint & [(\xi_1 - \eta_1) + (\cos\theta)(\xi_1 - \eta_1) + (\sin\theta)\zeta_1|\xi - \eta|] \\ & \times [\xi_2 + \eta_2 + (\cos\theta)(\xi_2 - \eta_2) + (\sin\theta)\zeta_2|\xi - \eta|]K(\theta)\,d\theta\,d\varphi \\ & - \int K(\theta)\,d\theta\,\xi_1\xi_2\end{aligned} \tag{99}$$

When multiplication of the first two factors in the integrand is carried out, followed by integration, all the terms involving ζ_i as well as $\cos\theta$ (not multiples of these) are going to drop out. So one need only compute

$$(1/2\pi)\tfrac{1}{4} \iint [(\xi_1 + \eta_1)(\xi_2 - \eta_2) + (\cos^2\theta)(\xi_1 - \eta_1)(\xi_2 - \eta_2) + (\sin^2\theta)\zeta_1\zeta_2|\xi - \eta|^2 K(\theta)\,d\theta\,d\varphi \tag{100}$$

The result will be a linear combination of $\xi_1\xi_2$, $\eta_1\eta_2$, $\xi_1\eta_2$, and $\eta_1\xi_2$. For the moment computation the first two terms are equivalent, as are the last two. It is necessary to write only one term of each pair. The factor $\xi_1\xi_2$ was already computed. It is $\lambda_{2,0} = \frac{1}{4}c_0 + \frac{3}{4}c_2$. As for the second factor, it is not difficult to show that

$$(1/2\pi)\int_{\zeta\perp\xi-\eta} \zeta_1\zeta_2\,d\varphi = -\tfrac{1}{2}(\xi_1 - \eta_1)(\xi_2 - \eta_2) \tag{101}$$

When this identity is used one gets $(\frac{1}{4}c_0 + \frac{3}{4}c_2)\xi_1\xi_2 + \frac{3}{4}(c_0 - c_2)\xi_1\eta_2$. When $c_0\xi_1\xi_2$ is substracted, one finally gets

$$T\xi_1\xi_2 = \tfrac{3}{4}(c_2 - c_0)(\xi_1\xi_2 - \xi_1\eta_2) \tag{102}$$

Similarly, by using (26) one gets

$$T\xi_1^2 = \tfrac{1}{4}(c_2 - c_0)\left(3\xi_1^2 - \sum \xi_k^2 - 2\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3\right) \tag{103}$$

Since $T\sum \xi_k^2 = 0$, one gets for the spherical moment $\phi_{2,0,1} \leftrightarrow 3\xi_1^2 - \sum \xi_k^2$ the result

$$T\left(3\xi_1^2 - \sum \xi_k^2\right) = \tfrac{3}{4}(c_2 - c_0)\left(3\xi_1^2 - \sum \xi_k^2 - 2\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3\right) \tag{104}$$

The next spherical moment will be a third-order one, $\phi_{3,1,1} \leftrightarrow \xi_1 \sum \xi_k^2$. In the computation the following identity is used:

$$\int_{\zeta \perp \xi - \eta} |\xi - \eta| \sum_k (\xi_k - \eta_k)\zeta_k \zeta_i \, d\varphi = 0, \qquad i = 1, 2, 3 \tag{105}$$

It follows that

$$T\xi_1 \sum \xi_k^2 = \tfrac{1}{2}(c_2 - c_0)\left(\xi_1 \sum \xi_k^2 - 2\xi_1 \sum \eta_1^2 + \xi_1 \sum \xi_k \eta_k\right) \tag{106}$$

and

$$T\xi_1^3 = \tfrac{3}{8}(c_2 - c_0)\left(3\xi_1^3 - \xi_1 \sum_k \xi_k^2 - 3\xi_1^2\eta_1 - \xi_1 \sum_k \eta_k^2 + 2\xi_1 \sum \xi_k \eta_k\right) \tag{107}$$

For the spherical moment $\phi_{3,0,1} \leftrightarrow 5\xi_1^3 - 3\xi_1 \sum \xi_k^2$

$$T\left(5\xi_1^3 - 3\xi_1 \sum \xi_k^2\right) = \tfrac{9}{8}(c_2 - c_0)\Big[\left(5\xi_1^3 - 3\xi_1 \sum \xi_k^2\right) - 5\xi_1^2\eta_1 + \xi_1 \sum_k \eta_k^2 + 2\xi_1 \sum \xi_k \eta_k\Big] \tag{108}$$

For the simplest spherical moment of order four, $\phi_{4,2}$,

$$T\left(\sum \xi_k^2\right)^2 = \tfrac{1}{2}(c_2 - c_0)\left[\left(\sum \xi_k^2\right)^2 - \sum \xi_k^2 \sum \eta_k^2 + \tfrac{1}{2} \sum_{i \neq j} (\xi_i \eta_j - \eta_i \xi_j)^2\right] \tag{109}$$

Let us exhibit the differential equations and the asymptotic formulas (67) for the following (unnormalized) spherical moments: $\phi_{2,0,1} \leftrightarrow \xi_1\xi_2$, $(\phi_{2,0,2} \leftrightarrow \xi_1\xi_3,\ \phi_{2,0,3} \leftrightarrow \xi_2\xi_3)$, $\phi_{2,0,4} \leftrightarrow 3\xi_1^2 - \sum \xi_k^2$, $(\phi_{2,0,5} \leftrightarrow 3\xi_2^2 - \sum \xi_k^2)$, $\phi_{3,1,1}$ $(\phi_{3,1,j} \leftrightarrow \xi_j \sum \xi_k^2)$, $\phi_{3,0,1}$ $(\phi_{3,0,i} \leftrightarrow 5\xi_i^3 - 3\xi_i \sum \xi_k)$.

For simplicity, denote first-order moments $\phi_i \leftrightarrow \xi_i$ and set $\lambda = c_2 - c_0$. We have

$$d\phi_{2,0,1}/dt = \tfrac{3}{4}\lambda(\phi_{2,0,1} - \phi_1\phi_2) \tag{110}$$

$$\phi_{2,0,1} - \phi_1\phi_2 = d_{2,0,1} \exp(\tfrac{3}{4}\lambda t) \tag{111}$$

$$d\phi_{2,0,4}/dt = \tfrac{3}{4}\lambda(\phi_{2,0,4} - 2\phi_1^2 + \phi_2^2 + \phi_3^2) \tag{112}$$

$$\phi_{2,0,4} - 2\phi_1^2 + \phi_2^2 + \phi_3^2 = d_{2,0,4} \exp(\tfrac{3}{4}\lambda t) \tag{113}$$

$$d\phi_{3,1,1}/dt = \tfrac{1}{2}\lambda(_{3,1,1} - \tfrac{5}{3}\phi_1\phi_{2,0} + \tfrac{1}{3}\phi_{2,0,4}\phi_1 + \phi_{2,0,1}\phi_2 + \phi_{2,0,2}\phi_3) \tag{114}$$

Substitute now the expressions obtained from (111) for $\phi_{2,0,1}$ and $\phi_{2,0,2}$ and the expression from (113) for $\phi_{2,0,4}$:

$$d\phi_{3,1,1}/dt + \tfrac{1}{2}\lambda\{\phi_{3,1,1} - \tfrac{5}{3}\phi_1\phi_{2,0} + \tfrac{1}{3}\phi_1(2\phi_1{}^2 - \phi_2{}^2 - \phi_3{}^2) + d_{2,0,4}\exp(\tfrac{3}{4}\lambda t) + \phi_2(\phi_1\phi_2 + d_{2,0,1}\exp(\tfrac{3}{4}\lambda t) + \phi_3[\phi_1\phi_3 + d_{2,0,2}\exp(\tfrac{3}{4}\lambda t)]\} \quad (115)$$

This we integrate

$$\phi_{3,1,1} - \tfrac{5}{3}\phi_1\phi_{2,0} = d_{3,1,1}\exp(\tfrac{1}{2}\lambda t) + (\tfrac{2}{3}\phi_1 d_{2,0,4} + 2\phi_2 d_{2,0,1} + 2\phi_3 d_{2,0,2})\exp(\tfrac{3}{4}\lambda t) \quad (116)$$

Observe that the second term on the right decays faster than the first one. Thus it has to be replaced by moments, using Eqs. (111) and (113) and resulting in an asymptotic expression for $\phi_{3,1,1}$ in terms of lower order moments:

$$\phi_{3,1,1} + \tfrac{1}{3}\phi_1\phi_{2,0} - \tfrac{2}{3}\phi_1\phi_{2,0,4} - 2\phi_2\phi_{2,0,1} - 2\phi_3\phi_{2,0,2} = d_{3,1,1}\exp(\tfrac{1}{2}\lambda t) \quad (117)$$

In an analogous way [from (108)]

$$d\phi_{3,0,1}/dt = \tfrac{9}{8}\lambda(\phi_{3,0,1} - \phi_1\phi_{2,0,4} + 2\phi_{2,0,1}\phi_2 + 2\phi_{2,0,2}\phi_3) \quad (118)$$

$$\phi_{3,0,1} - 3\phi_1\phi_{2,0,4} + 6\phi_{2,0,1}\phi_2 + 6\phi_{2,0,2}\phi_3 - 6\phi_1\phi_2{}^2 - 6\phi_1\phi_3{}^2 + 4\phi_1{}^3 = d_{3,0,1}\exp(\tfrac{9}{8}\lambda t) \quad (119)$$

Unfortunately, the hydrodynamic equations are expressed in terms of ordinary moments, so one has to express the latter in terms of spherical moments. We will do this with a "hydrodynamic notation." Let us define the thermohydrodynamic variables in correspondence with those used in this paper. The variables below depend, of course, on x and t:

$$\rho \leftrightarrow \psi_{(0,0,0)} \quad (120)$$

$$u_i,\ i = 1, 2, 3: \qquad \rho u_1 \leftrightarrow \psi_{(1,0,0)} \leftrightarrow \phi_1 \leftrightarrow \xi_1 \quad (121)$$

$$E: \qquad \rho E \leftrightarrow \tfrac{1}{2}(\psi_{(2,0,0)} + \psi_{(0,2,0)} + \psi_{(0,0,2)}) \leftrightarrow \tfrac{1}{2}\phi_{2,0} \leftrightarrow \tfrac{1}{2}\sum_k \xi_k{}^2 \quad (122)$$

$$T_{ij}: \qquad \rho T_{12} \leftrightarrow \psi_{(1,1,0)} \leftrightarrow \phi_{2,0,1} \leftrightarrow \xi_1\xi_2 \quad (123)$$

$$\rho(3T_{11} - E) \leftrightarrow 2\psi_{(2,0,0)} + \psi_{(0,2,0)} + \psi_{(0,0,2)} \leftrightarrow \phi_{(2,0,4)} \leftrightarrow 3\xi_1{}^2 - \sum_k \xi_k{}^2 \quad (124)$$

$$P_{ijk}: \qquad \rho P_{111} \leftrightarrow \psi_{(3,0,0)} \leftrightarrow \xi_1{}^3 \leftrightarrow \tfrac{1}{5}\left[\left(5\xi_1{}^3 - 3\xi_1\sum \xi_k{}^2\right) + 3\xi_1\sum \xi_k{}^2\right] \leftrightarrow \tfrac{1}{5}(\phi_{3,0,1} - 3\phi_{3,1,1}) \quad (125)$$

$$R_i: \qquad \rho R_1 \leftrightarrow \tfrac{1}{2}\xi_1\sum \xi_k{}^2 \leftrightarrow \tfrac{1}{2}(\psi_{(3,0,0)} + \psi_{(1,2,0)} + \psi_{(1,0,2)}) \leftrightarrow \tfrac{1}{2}\phi_{3,1,1} \quad (126)$$

The equations (90) for the moments up to order two will be

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial}{\partial x_i} \rho u_i = 0 \tag{127}$$

$$\frac{\partial \rho u_j}{\partial t} + \sum_i \frac{\partial}{\partial x_i} \rho T_{ij} = 0 \tag{128}$$

By (90) and (102)

$$\frac{\partial \rho T_{jk}}{\partial t} + \sum_i \frac{\partial}{\partial x_i} \rho P_{ijk} = \frac{3}{4} \lambda \rho^2 (T_{jk} - u_j u_k), \qquad j \neq k \tag{129}$$

By (90) and (103)

$$\frac{\partial \rho T_{jj}}{\partial t} + \sum_i \frac{\partial}{\partial x_i} \rho P_{ijj} = \frac{3}{4} \lambda \rho^2 \left(T_{jj} - \frac{2}{3} E - u_j^2 + \frac{1}{3} \sum_i u_k^2 \right) \tag{130}$$

In particular

$$\frac{\partial \rho E}{\partial t} + \sum_i \frac{\partial}{\partial x_i} \rho R_i = 0 \tag{131}$$

The Euler equations are derived by inserting the absolute limits for T_{ij} from (111) and (113) (i.e., the right-hand side in these equations is taken to be zero). The limiting values of R_i are obtained from (116). Equation (127) remains unchanged. Equation (128) minus (127) multiplied by u_j becomes

$$\frac{\partial u_j}{\partial t} + \sum_i u_i \frac{\partial u_j}{\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \tag{132}$$

where $p = \frac{1}{3}\rho(2E - \sum_k u_k^2) = \frac{1}{3}\rho T$. Thus T is indeed some normalized temperature.

Equation (131) with the limit (116) becomes

$$\frac{\partial \rho E}{\partial t} + \frac{5}{3} \sum \frac{\partial}{\partial x_i} \rho u_i E = 0 \tag{133}$$

In the second approximation, Eqs. (127) and (132) are unchanged. Equation (131) holds (R_i are now free variables) and four additional equations for R_i and the fourth-order moment $\phi_{4,2}$ have to be added.

The third approximation restricted to moments of order two consists of Eqs. (127)–(130), where in the last two equations one has to substitute for ρP_{ijk} using the asymptotic formulas (117) and (119). For this it is enough to know the asymptotic formulas for ρP_{111}, ρP_{112}, and ρP_{123}. The equation for ρP_{111} is obtained via (125) by the difference of Eqs. (117) and (119):

$$\begin{aligned} \rho P_{111} \to \tfrac{1}{5}\rho^2[3u_1(3T_{11} - 2E) - 6T_{12}u_2 \\ - 6T_{13}u_3 + 6u_1u_2^2 + 6u_1u_3^2 - uu_1^3] + \tfrac{6}{5}\rho R_1 \end{aligned} \tag{134}$$

The computation of ρP_{112} is simpler if invariance (under rotations) arguments are used. Consider the two rotations U and U^{-1}:

$$U:\quad \xi_1 \to (1/\sqrt{2})(\xi_1 + \xi_2), \quad \xi_2 \to (1/\sqrt{2})(\xi_2 - \xi_1)$$

Express $\rho P_{112} \leftrightarrow \xi_1{}^2\xi_2$ by

$$3\xi_1{}^2\xi_2 = \sqrt{2}\{[(1/\sqrt{2})(\xi_1 + \xi_2)]^3 - [(1/\sqrt{2})(\xi_1 - \xi_2)]^3\} - \xi_2{}^3$$

Perform the rotations on (134) and compute the combination [under U: $u_1 \to (1/\sqrt{2})(u_1 + u_2)$, $T_{11} \to \frac{1}{2}(T_{11} + T_{22} + 2T_{12})$, u_3 remains unaltered, and so on]. The result is

$$\rho P_{112} \to \tfrac{1}{5}\rho^2[u_2(5T_{11} - 2T_{22} - E) + 8u_1T_{12} - 2u_3T_{23} + 2u_2{}^2 + 2u_2u_3{}^2 - 8u_1{}^2u_2] + \tfrac{3}{5}\rho R_2 \tag{135}$$

Operate on the terms in (135) by rotating by V and V^{-1}, where

$$V^{-1}:\quad \xi_1 \to (1/\sqrt{2})(\xi_1 + \xi_3), \quad \xi_3 \to (1/\sqrt{2})(\xi_3 - \xi_1)$$

Then

$$\xi_1\xi_2\xi_3 = \tfrac{1}{2}\xi_2[(1/\sqrt{2})(\xi_1 + \xi_3)]^2 - [(1/\sqrt{2})(\xi_3 - \xi_1)]^2$$

Hence

$$\rho P_{123} \leftrightarrow \rho^2(u_1T_{23} + u_2T_{13} + u_3T_{12} - 2u_1u_2u_3) \tag{136}$$

In order to complete the derivation, one has to insert (134)–(136) into the appropriate formulas.

It seems that the resulting formulas are too complicated to be treated explicitly and a call for a general approach is in order.

Let us conclude this section with some remarks pertaining to kernels having infinite cross section.

The dependence of the hydrodynamic equations (110)–(130) on the explicit form of the collision kernel is via the constant

$$\lambda = c_2 - c_0 = \int (\cos^2\theta - 1)K(\theta)\, d\theta \tag{137}$$

λ is finite for some kernels having a singularity at $\theta = 0$. For λ to be finite, $K(\theta)$ may have, near zero, a rate of growth of the form $\theta^{-3+\epsilon}$.

Let us define $K_N(\theta)$ by

$$\begin{aligned} K_N(\theta) &= K(\theta), &\quad |K(\theta)| &\leqslant N \\ K_N(\theta) &= N, &\quad K(\theta) &\geqslant N \end{aligned} \tag{138}$$

and

$$\lambda_N = \int (\cos^2 \theta - 1) K_N(\theta)\, d\theta$$

Then, clearly, $\lambda_N \to \lambda$.

Equations (110)–(130) consist of two parts. Equations (110)–(119) exhibit asymptotic formulas that are valid for the spatially homogeneous case. Equations (120)–(130) exhibit the thermohydrodynamic equations suggested for the spatially inhomogeneous case. Equations (110)–(119) are continuous with respect to the parameter λ; therefore they are well defined for $K(\theta)$. In this way, kernels having a singularity at $\theta = 0$, as specified above, can be treated. The Maxwellian kernel falls into this category, as is shown in the appendix.

APPENDIX. ESTIMATE OF THE SINGULARITY OF MAXWELL'S KERNEL

For $n = 5$, Eq. (9) of Ref. 12 (Vol. 2, p. 40) reads

$$\frac{\pi}{2} - \theta = \int_0^{x'} \frac{dx}{[1 - x^2 - c(x/b)^4]^{1/2}} \tag{A.1}$$

where x' is the first zero of the denominator (if $n = 5$, the outcome is independent of V; thus one may take $V = 1$ and get $\alpha = c_0 b$). Equation (A.1) serves as an (implicit) definition of θ as a function of b.

Now $K(\theta)$ is defined by

$$\int_{\theta_1}^{\theta_0} K(\theta)\, d\theta = -\int_{\theta_0}^{\theta_1} K(\theta)\, d\theta = \int_{b_0}^{b_1} \hat{K}\, db \tag{A.2}$$

where $0 \leqslant b < \infty$, $0 \leqslant \theta \leqslant \pi/2$, $\theta_i = \theta(b_i)$, $i = 0, 1$, and $\hat{K}$ is a constant ($n = 5$), which depends on mass, density, etc. Thus

$$K(\theta) = -\frac{db}{d\theta} \hat{K} \tag{A.3}$$

For $b = \infty$, $x' = 1$. The right-hand side of (A.1) is equal to $\pi/2$, i.e., $\theta = 0$. Denote, for convenience, $y = x'$ and study the dependence of y on b:

$$1 - y^2 - \frac{cy^4}{b^4} = 0$$

Differentiate:

$$-\frac{dy}{db}\left(2y + \frac{4cy^3}{b^4}\right) = \frac{4cy^4}{b^5}$$

Thus

$$y = 1 - \frac{c}{2b^4} + \frac{c^2}{b^8} - \cdots \tag{A.4}$$

$$\frac{dy}{db} = \frac{2c}{b^5} - \frac{8c^2}{b^9} + \cdots \tag{A.5}$$

Thus, for large b

$$y \sim 1 - \frac{c}{2b^4} = z \tag{A.6}$$

Define $\varphi(b)$ by

$$\frac{\pi}{2} - \varphi(b) = \int_0^z \frac{dx}{(1 - x^2)^{1/2}} \tag{A.7}$$

It follows that $\varphi(b) \sim \theta(b)$. It is not difficult to see that also $\varphi'(b) \sim \theta'(b)$ [$\varphi(\infty) = \theta(\infty) = 0$ and the reduction is done in a "consistent" manner; the suspicious reader may verify it by another expansion]. Thus

$$\hat{K}\frac{db}{d\varphi} \sim \hat{K}\frac{db}{d\theta} = K(\theta) \tag{A.8}$$

Integrate (A.7) to get

$$\frac{\pi}{2} - \varphi(b) = \sin^{-1}\left(1 - \frac{c}{2b^4}\right), \qquad \varphi^2(b) \sim \frac{c}{b^4}$$

$$-\frac{d\varphi}{db} - \frac{\sqrt{c}}{b^3}$$

$$-\frac{db}{d\varphi} \sim \frac{b^3}{\sqrt{c}} \sim \varphi^{3/2}\hat{c} \tag{A.9}$$

Thus, taking (A.8) into account,

$$K(\theta) \sim \theta^{-3/2} \times \text{const}$$

near $\theta = 0$.

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REFERENCES

1. S. G. Brush, Interatomic Forces and Gas Theory from Newton to Lennard-Jones, *Arch. Rat. Mech. Anal.* **39**:1 (1970).
2. T. Carleman, *Problemes Mathematiques dans la Theorie Cinetique des Gas*, Publications Scientifiques de l'Institut Mittag-Leffler, Vol. 2, Uppsala (1957).
3. S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press (1939).
4. G. Ford and G. Uhlenbeck, *Lectures in Statistical Mechanics*, Am. Math. Soc. (1963).
5. I. N. Gelfand, R. A. Minlos, and Z. Y. Shapiro, *Representations of the Rotation and Lorentz Group and their Applications*, Pergamon Press, Oxford (1963).
6. H. Grad, On the Kinetic Theory of Rarified Gases, *Commun. Pure Appl. Math.* **2**:331 (1949).
7. H. Grad, Asymptotic Theory of the Boltzmann Equation, in *Rarified Gas Dynamics*, J. A. Laurmann, ed., Academic Press (1963), Vol. 1, pp. 26–60.
8. H. Grad, Asymptotic Equivalence of the Navier–Stokes and Non-Linear Boltzmann Equations, in *Symposia in Applied Mathematics*, Vol. 17, Am. Math. Soc. (1965).
9. H. Grad, Solution of the Boltzmann Equation in Unbounded Domain, *Commun. Pure Appl. Math.* **18**:345 (1965).
10. E. Ikenberry and C. Truesdell, On the Pressures and the Flux of Energy in a Gas According to Maxwell's Kinetic Theory. I. *J. Rat. Mech. Anal.* **5**:1 (1956).
11. C. Truesdell, On the Pressures and the Flux of Energy in a Gas According to Maxwell's Kinetic Theory. II. *J. Rat. Mech. Anal.* **5**:55 (1956).
12. J. C. Maxwell, *The Scientific Papers of James Clerk Maxwell*, Cambridge University Press (1890)
13. O. E. Lanford, *Time Evolution of Large Classical Systems*, Lecture Notes in Physics, No. 38, J. Moser, ed., Springer-Verlag (1974).
14. C. Truesdell, *Rational Thermodynamics*, McGraw-Hill (1969), 8th Lecture.
15. L. Arkeryd, On the Boltzmann Equation. Part I. *Arch. Rat. Mech. Anal.* **40**:1 (1972).
16. L. Arkeryd, On the Boltzmann Equation. Part II. *Arch. Rat. Mech. Anal.* **40**:17 (1972).
17. H. Grad, Thermodynamic des Gase, *Handbuch der Physik*, Vol. 12, Springer (1958).